Signal & Data Analysis in Neuroscience 2020

The Frequency Domain

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Outline

- Introduction
- Fourier Transform
- Sampling Theory
- Systems
- Filtering
- Spectral Analysis

Large parts of the lecture were built by **Iddo Lev**

Based on: **Signals & systems** / Oppenheim, Willsky & Nawab
Discrete signal processing / Oppenheim & Schafer

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Periodic signal

- A *periodic signal* repeats its values after some definite period has been added to its independent variable.
- Everyday examples are seen when the variable is time:
  - Hands of a clock
  - Phases of the moon

$s(x+p)=s(x)$ for the cycle period p

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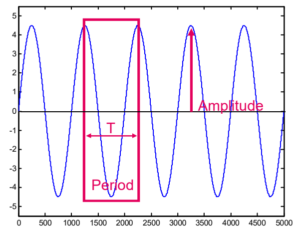
Sinusoidal signal
Frequency

$$x(t) = A\sin(\omega t)$$



A = 4.5

T = 1000

f=1/T

(1/1000)

$\omega = 2\pi f$

($2\pi/1000$)

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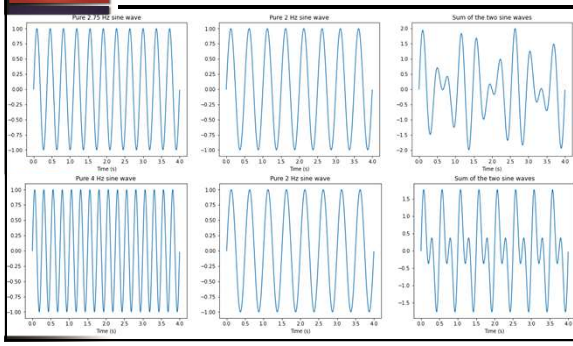
Sinusoidal signal
Sum of two signal (different frequencies)



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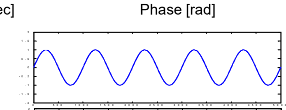
Sinusoidal signal
Frequency and Phase

$$x(t) = A\sin(\omega t + \phi)$$

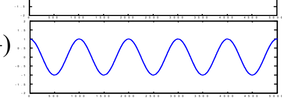
Frequency [rad/sec]

Phase [rad]

$x(t) = A\sin(\omega t)$



$x(t) = A\sin(\omega t + \frac{\pi}{2})$



Remember: 2π radians are equal to 360 degrees…

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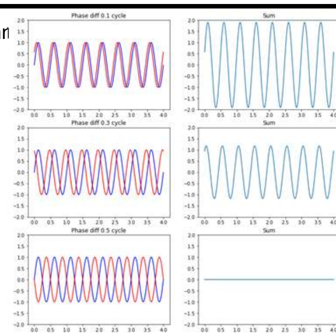
Sinusoidal signals
Sum of two signal (different phases)

Sumn ... $_A = \omega_B$)



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Sinusoidal signals
Example: auditory signals

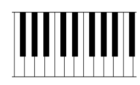
Sound waves:

Different tones –
Different frequencies

2000 Hz

1000 Hz

500 Hz



A	220 Hz
A#	233 Hz
B	247 Hz
C	262 Hz
C#	277 Hz
D	294 Hz
D#	311 Hz
E	330 Hz
F	349 Hz
F#	370 Hz
G	392 Hz
G#	415 Hz
A	440 Hz

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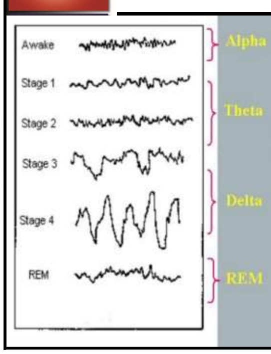
Example: EEG during sleep



State	Frequency range	State of mind
Delta	0.5 – 4 Hz	Deep sleep
Theta	4 – 8 Hz	Drowsiness (also first stage of sleep)
Alpha	8 – 14 Hz	Relaxed but alert
Beta	14 – 30 Hz	Highly alert and focused

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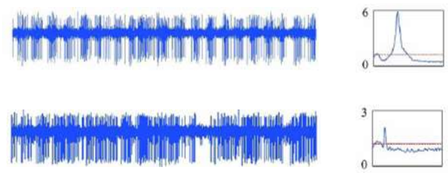
Example: Neuronal activity in Parkinson's disease patients

Parkinsonian Tremor : Globus Pallidus activity



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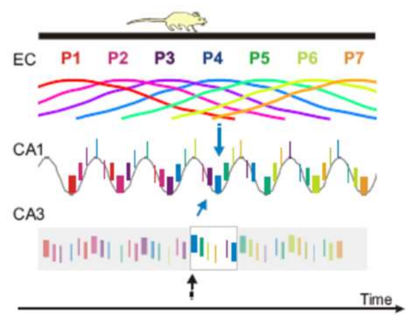
Example: Theta rhythm and place cells in the hippocampus



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Fourier explained…

https://www.youtube.com/watch?v=spUNpyF58BY

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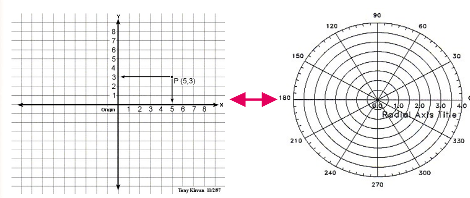
Transform example - Cartesian vs. Polar coordinates



$$x = r\cos\theta \qquad y = r\sin\theta \quad \longleftrightarrow \quad r = \sqrt{x^2+y^2} \qquad \theta = \arctan\frac{y}{x} \quad x \neq 0$$

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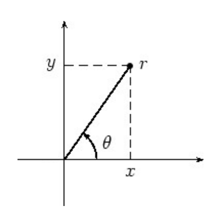
Transform example - Cartesian vs. Polar coordinates

Completely interchangeable → no information loss



$(x,y) \longleftrightarrow (r,\theta)$

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Complex Numbers

- **Imaginary number**: $\sqrt{-1} = i\ (or\ j)$ $\qquad i^2 = -1$
- i (or j) = 1 unit of imaginary number (like 1 in real numbers)
- A combination of a real number with an imaginary number is called a **complex number**
- Complex numbers, e.g: α ± iβ, 5+3i, 3.24 – 5.88i, ….
- Complex Conjugated couple := α + iβ, α – iβ
- Ā,A* = conjugated of A (A = α + iβ ; A*= α – iβ)

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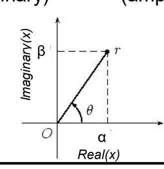
The complex plane

- X axis → Real part of the number
- Y axis → Imaginary part of the number

C(α, β) = α + iβ ⟷ C(r, θ)

(real, imaginary) ⟷ (amplitude, phase)



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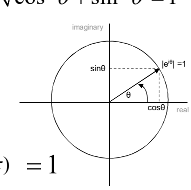
Complex Exponentials

- Definition: $e^{i\theta} = \cos\theta + i\sin\theta$
- $\left|e^{i\theta}\right| = \left|\cos\theta + i\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$
- Unit Circle:



- $e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$

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Exponentials ←→ Trigonometric functions

$$e^{i\theta} = \cos\theta + i\cdot\sin\theta \qquad e^{-i\theta} = \cos\theta - i\cdot\sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} \qquad \cot\theta = i\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}$$

$$e^{i(\theta + 2k\pi)} = e^{i\theta} \qquad k = \text{integer}$$

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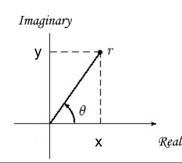
Complex representation

- X axis → Real part of the number
- Y axis → Imaginary part of the number

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



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Fourier Transform

Jean Baptiste Joseph Fourier

Time domain x(t) $\overset{\mathcal{F}}{\longleftrightarrow}$ Frequency Domain X(ω)

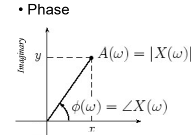
Typically real number

(can be complex as well)

- Amplitude: voltage, height, length, …

Complex number

- Amplitude = ~Energy
- Phase



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Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$e^{inx} = \cos(nx) + i\sin(nx)$$

$$F_n = (a_n - ib_n)/2 \quad , \quad F_n = F_{-n}^*$$

$$\rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} F_n e^{inx}$$

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Fourier Transform

Notice : infinity limits

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{inx}$$

x → ω
f → X
n → -t
F_n → x(t)

x → t
f → x
n → ω
F_n → X(ω)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$$

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\,d\omega$$

Inverse Fourier Transform

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Continuous Fourier Transform

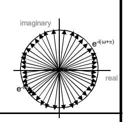
$Example: \quad x(t) = \sin(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \sin(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\left(\frac{e^{it} - e^{-it}}{2i}\right)e^{-i\omega t}dt$$

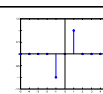
$$= \frac{1}{2i}\int_{-\infty}^{\infty}(e^{it} - e^{-it})e^{-i\omega t}dt = \frac{1}{2i}\int_{-\infty}^{\infty} e^{it}e^{-i\omega t}dt - \frac{1}{2i}\int_{-\infty}^{\infty} e^{-it}e^{-i\omega t}dt$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} e^{-i(\omega-1)t}dt - \frac{1}{2i}\int_{-\infty}^{\infty} e^{-i(\omega+1)t}dt$$

$$= \frac{1}{2i}\delta(\omega - 1) - \frac{1}{2i}\delta(\omega + 1)$$



$$X(\omega) = \begin{cases} \frac{1}{2i}\infty & \omega = 1 \\ -\frac{1}{2i}\infty & \omega = -1 \\ 0 & \forall \omega \neq 1, -1 \end{cases}$$



(Normalized)

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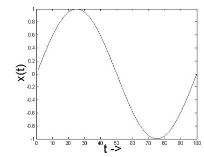
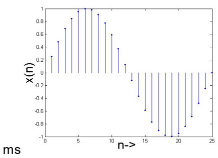
Digital Signals

- Continuous function : x(t)



- Discrete function : x[n]

T_s: sample rate, $x[n] = x(nT_s)$



$T_s = 4$ ms

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Discrete Fourier Transform (DFT)

- Discrete both in time (x[n]) and frequency (X[k])
- Finite signal length →

  Only N samples = x[1],x[2],………,x[N]

↓

“Discretizing” the frequency domain

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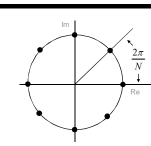
Discretizing the frequency domain

Frequency described by N equally spaced values:



$$\omega \rightarrow \left\{\frac{2\pi k}{N}\right\}_{k=0}^{N-1} \quad \longrightarrow \quad e^{-i\omega} \rightarrow \left\{e^{-i\frac{2\pi k}{N}}\right\}_{k=0}^{N-1}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt \quad \longrightarrow \quad \boxed{X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \quad for\ k=0,\dots,N-1}$$

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DFT - formulation

Discrete Fourier Transform (DFT)

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn} \qquad k = 0, \dots, N-1$$

Inverse Discrete Fourier Transform (IDFT)

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} \qquad n = 0, \dots, N-1.$$

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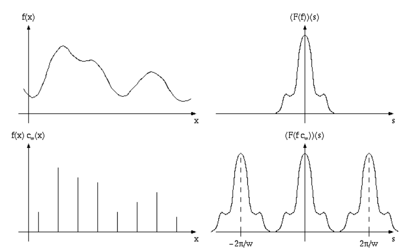
DFT – Periodic in frequency

$$e^{-i\frac{2\pi k}{N}(n+N)} = e^{-i\frac{2\pi k}{N}n} e^{-i\frac{2\pi k}{N}N} = e^{-i\frac{2\pi k}{N}n} (e^{-i2\pi})^k = e^{-i\frac{2\pi k}{N}n}$$



Oliver Kreylos, ucdavis.edu

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Discrete Fourier Transform

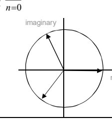
$$\textit{Example:} \qquad x[n] = \sin(\frac{2\pi}{N}n)$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{2\pi i}{N}kn} = \sum_{n=0}^{N-1} \sin(\frac{2\pi}{N}n) e^{-\frac{2\pi i}{N}kn} = \sum_{n=0}^{N-1} \frac{e^{i\frac{2\pi}{N}n} - e^{-i\frac{2\pi}{N}n}}{2i} e^{-\frac{2\pi i}{N}kn}$$

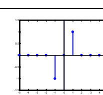
$$= \frac{1}{2i} \sum_{n=0}^{N-1} (e^{i\frac{2\pi}{N}n} - e^{-i\frac{2\pi}{N}n}) e^{-\frac{2\pi i}{N}kn} = \frac{1}{2i} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} e^{-\frac{2\pi i}{N}kn} - \frac{1}{2i} \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}n} e^{-\frac{2\pi i}{N}kn}$$

$$= \frac{1}{2i} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(k-1)n} - \frac{1}{2i} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(k+1)n}$$

$$= \frac{1}{2i} N\delta[k-1] - \frac{1}{2i} N\delta[k+1]$$



$$X[k] = \begin{cases} \frac{1}{2i}N & k = 1 \\ -\frac{1}{2i}N & k = -1 \ [\rightarrow k = N-1] \\ 0 & \forall k \neq 1, -1 \end{cases}$$



(Normalization)

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FFT – Fast Fourier Transform

An **efficient algorithm** to compute the DFT and its inverse :

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-\frac{2\pi i}{N}nk}$$ N times k * N times n

FFT: **O(n^2) → O(n log(n))**

(Based on $X[n-k]=X^*[k]$, for real signals)

Frequently confused as a method for spectral evaluation. Wrong !!!

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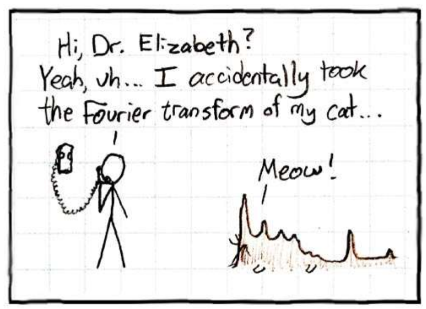
You're not the only ones going crazy with Fourier transforms



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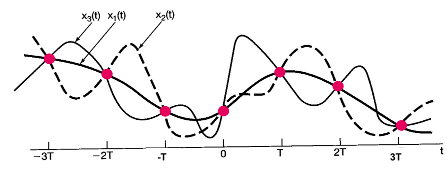
Sampling : Analog to Digital

- Creating x[n] from x(t)
- T_s – interval between samples, $X[n] = x(nT_s)$



Signals & Systems, Oppenheim

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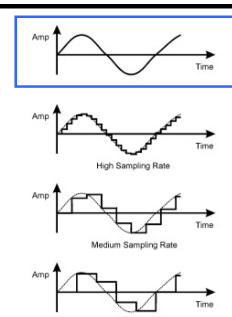
Sampling Frequency

$f_s = 1/T_s$

Periods in DFT – Every f_s



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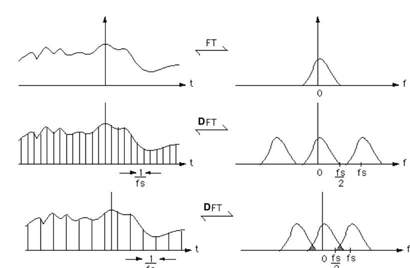
Aliasing



Paul Bourke

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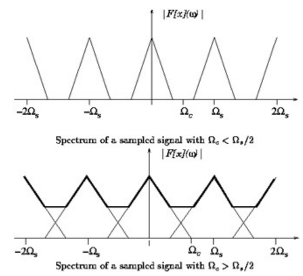
Aliasing Issues



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Sampling Theorem

Nyquist H., "Certain Topics in Telegraph Transmission Theory", 1928

- Let x(t) be band-limited signal with $X(\omega) = 0$ for $|\omega|>\omega_M$.
- Then x(t) is uniquely determined by its samples x(nT), n=0,±1,±2…. if $\omega_s > 2\omega_M$

Given these samples, x(t) can be recovered exactly.

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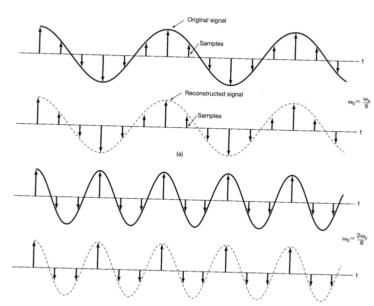
Correct Sampling = $\omega_s > 2\ \omega_0$



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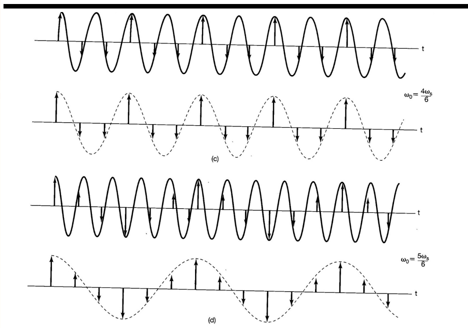
Wrong Sampling - $\omega_s < 2\,\omega_0$



Signals & Systems, Oppenheim

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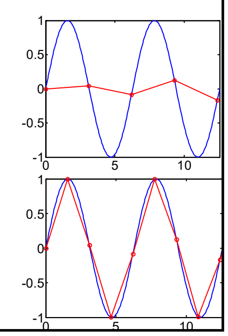
Practical sampling

- Nyquist rule:
  $f_s \geq 2 * \max(f)$

- In practice:
  f_s is 3-4 times max(f)



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So what do we do with all that?

- Find out what frequencies we have in the signal
- Separate different processes by their frequencies
- Discover oscillations
- Reveal how the frequencies, and therefore processes, change over time

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