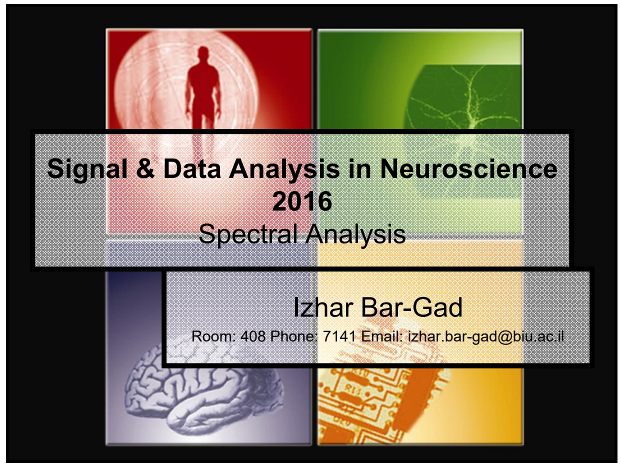

Outline – Frequency domain

- ☑ Introduction
- ☑ Fourier Transform
- ☑ Sampling Theory
- ☑ Systems
- ☑ Filters
- Spectral Analysis

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Correlation & Convolution

- Correlation

$$R_{g,h}(t) = Corr(g,h)_t = \sum_{i=-\infty}^{\infty} g_i h_{t+i}$$

$$R_{g,h}(t) = Corr(g,h)_t = Corr(h,g)_{-t} = R_{h,g}(-t)$$

- Convolution

$$(g*h)_t = \sum_{i=-\infty}^{\infty} g_{t-i} h_i$$

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Parseval's theorem

The Fourier transform is unitary → the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

Continuous Fourier transform

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Discrete Fourier transform

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$$

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Signal energy

- The energy spectral density describes how the energy (or variance) of a signal or a time series is distributed with frequency.

$$\Phi(\omega) = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \right|^2 = \frac{F(\omega) F^*(\omega)}{2\pi}$$

$$\Phi(\omega) = \left| \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f_n e^{-i\omega n} \right|^2 = \frac{F(\omega) F^*(\omega)}{2\pi}$$

(only for finite energy signals)

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Signal Power

$$Power = \frac{Energy}{Time}$$

$$S = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$S = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

Equal By Parseval's Theorem

$$\hat{S}[f] = |DFT\{x[n]\}|^2 = \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{\left(\frac{-j2\pi nf}{N}\right)} \right|^2$$

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Wiener-Khinchin theorem

- The power spectrum is the Fourier transform of the auto-correlation function
- Power spectrum $S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f\tau}\, d\tau$
- Autocorrelation $R(\tau) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau}\, df$

Power spectral density

- Amount of power per unit (density) of frequency (spectral) as a function of the frequency

Power Spectrum	Spectral Density
$S_{x,x}(\omega) = \sum_{m=-\infty}^{\infty} R_{x,x}(m) \cdot e^{-j\omega m}$	$P_{x,x}(\omega) = \frac{S_{x,x}(\omega)}{2\pi}$
$S_{x,x}(f) = \sum_{m=-\infty}^{\infty} R_{x,x}(m) \cdot e^{\frac{-2\pi fm}{f_s}}$	$P_{x,x}(f) = \frac{S_{x,x}(f)}{f_s}$

Average power

$$\overline{P}(\omega_1, \omega_2) = \int_{\omega_1}^{\omega_2} P_{x,x}(\omega) d\omega - \int_{-\omega_2}^{-\omega_1} P_{x,x}(\omega) d\omega = 2 \cdot \int_{\omega_1}^{\omega_2} P_{x,x}(\omega) d\omega$$

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Spectrum Estimation - Problems

- Leakage problems and side lobes.
- Increased length of signal leads to increase in number of discrete frequency but not to increased accuracy at each frequency.

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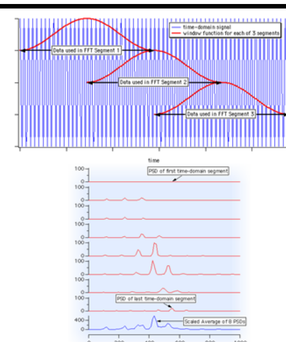
Spectrum Estimation Methods

- Non parametric estimators
- Correlogram estimators
- Parametric estimators
- Subspace estimators

Example – Welch estimator

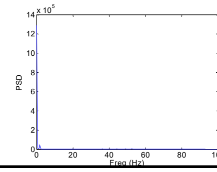


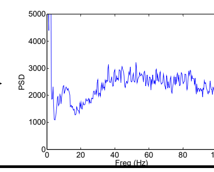
Things we find in the spectrum:

- DC
- White noise
- Harmonics

Direct Current (DC)

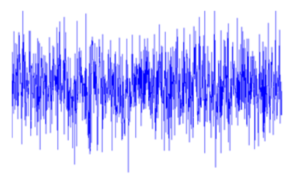
- DC is transformed into energy at frequency 0.
- Unless DC is removed the power it adds is typically huge compared to all other frequencies.





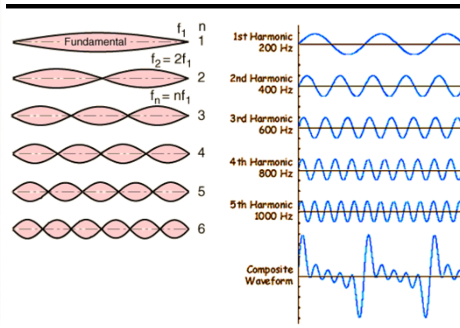
White noise

- White noise is a random signal (or process) with a flat power spectral density.



- Zero correlation at t≠0.
- There are also pink & red/brown noises.

Harmonics



Do we have harmonies?

- Many biological processes lead to the formation of harmonies.
- Any square wave (for example a sudden change in a parameter) is transformed to multiple harmonies.
- The multiple frequencies may describe the same underlying process.

Temporal & spectral resolution

- Using **windowed estimation** (Welch/Bartlett) leads to a temporal / spectral resolution tradeoff.
- For a recording of ***T*** seconds sampled at ***s*** samples/sec and assessed using a ***w*** sample window:

Number of windows: $\frac{T \cdot s}{w}$

Spectral resolution (Δf): $\frac{s}{w}$

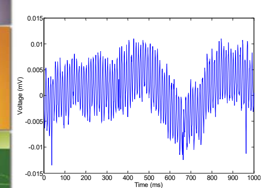
Relative & absolute power

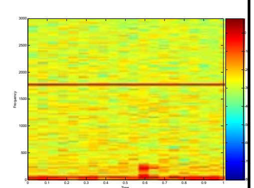
- The absolute power depends heavily on the normalization of the signal.
- The relative power enables detecting the statistics of the signal at unfiltered frequencies.

Spectogram

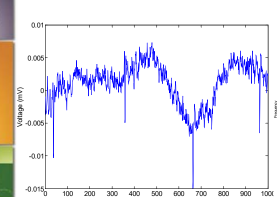
- In many cases the signal is not stationary.
- However, assuming stationarity over short intervals leads to usage of the spectrogram.
- Power spectral density over short periods of time using a sliding window over the signal
- Temporal resolution vs. spectral resolution…

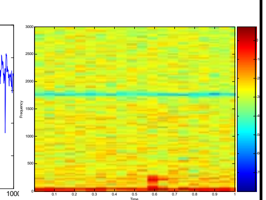
Original signal



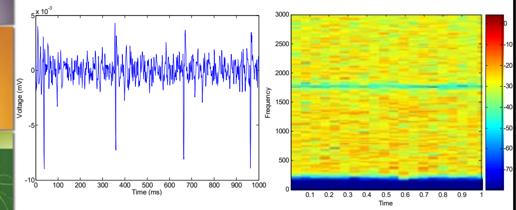


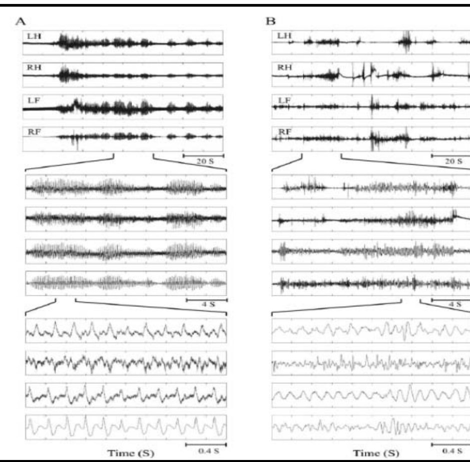
Following notch filter (~1800Hz)

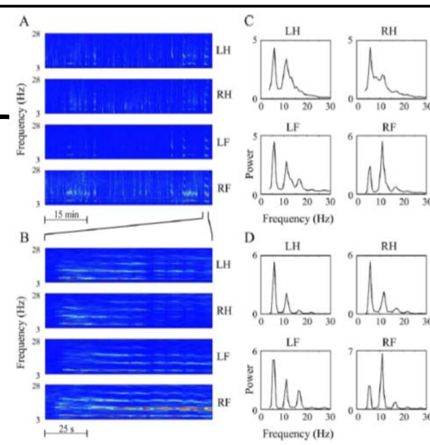




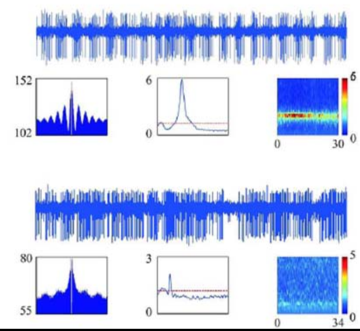
Following high pass (~300Hz)







GP activity - Parkinsonian primate



Cross spectrum I

- Multiplication of the Fourier transforms of the signals.

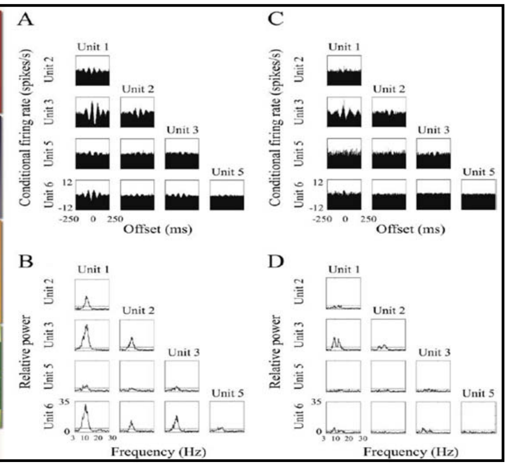
$$\hat{S}_{y,x}(\omega) = \frac{1}{N} Y(\omega) X^*(\omega)$$

Cross spectrum II

- Cross Spectral Density – defined as the transform of the cross correlation

$$S_{x,y}(\omega) = \sum_{m=-\infty}^{\infty} R_{x,y}(m) \cdot e^{-j\omega m}$$

- Two unrelated signals sharing common frequencies will have significant cross spectral density over finite length of time.



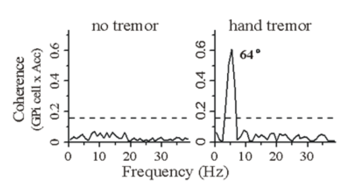
Coherence

- Coherence - ratio of the cross spectral density to the power spectral density of the two signals

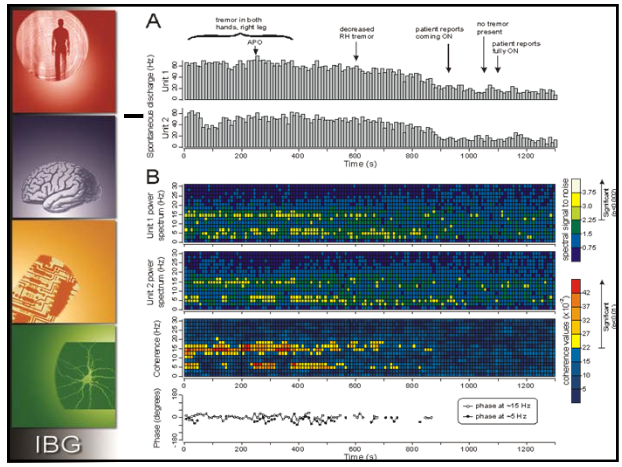
$$C_{x,y}(f) = \frac{\left|S_{x,y}(f)\right|^2}{S_{x,x}(f) \cdot S_{y,y}(f)}$$

- Normalizes to the spectrum of the two signals and thus relates only to the relation between the signals and not to their structure.

Coherence example



(Taken from Dostrovsky et al)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A
tremor in both hands, right leg
APO
decreased RH tremor
patient reports coming ON
no tremor present
patient reports fully ON
Spontaneous discharge (Hz)
Unit 1
Unit 2
Time (s)
B
Unit 1 power spectrum (Hz)
Unit 2 power spectrum (Hz)
Coherence (Hz)
spectral signal to noise
coherence values
Phase (degrees)
phase at ~15 Hz
phase at ~5 Hz
Time (s)