SIGNAL & DATA ANALYSIS IN NEUROSCIENCE 2020

FREQUENCY DOMAIN

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Outline

- Fourier transform
- Sampling theorem + aliasing
- Systems

Fourier transform (FT)

- Fourier Transform – transforms information between time domain and frequency domain.
- The continuous FT:

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot exp^{-i\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) \cdot exp^{i\omega t} d\omega$$

- The DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot exp^{-i\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot exp^{i\frac{2\pi kn}{N}}$$

- The output of FT is a representation of the signal by frequency components.

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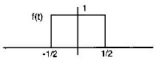
Example: the Fourier Transform of a rectangle function: rect(t)

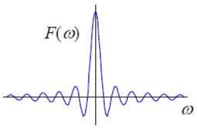


$$F(\omega) = \int_{-1/2}^{1/2} e^{-i\omega t}\,dt = \frac{1}{-i\omega}[e^{-i\omega t}]_{-1/2}^{1/2}$$

$$= \frac{1}{-i\omega}[e^{-i\omega/2} - e^{i\omega/2}]$$

$$= \frac{1}{(\omega/2)} \frac{e^{i\omega/2} - e^{-i\omega/2}}{2i}$$

$$= \frac{\sin(\omega/2)}{(\omega/2)}$$



$$\boxed{F(\omega) = \mathrm{sinc}(\omega/2)}$$

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Some useful time ←→ frequency pairs

TABLE A.2 Fourier-Transform Pairs.

Time Function	Fourier Transform
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\mathrm{sinc}(fT)$
$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \ge T \end{cases}$	$T\mathrm{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$g(t-t_0)$	$\exp(j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$

Notes: $\delta(t)$ = delta function, or unit impulse
rect(t) = rectangular function of unit amplitude and unit duration centered on the origin
sinc(t) = sinc function

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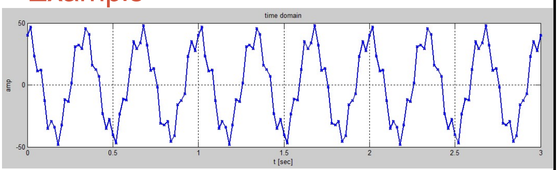
Example



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Some Terms

- Power = $amplitude^2$ (by definition)
- Decibel (dB) is a measure of the ratio between two quantities. For our uses it usually measures power:
  - $10*\log10(Power_1/Power_0)$ =
    $10*\log10(amp_1^2/amp_0^2)$ =
    $10*\log10[(amp_1/amp_0)^2]$ =
    $20*\log10(amp_1/amp_0)$
- Matlab functions:
  - fft(x) FFT for x result $[0, 2\pi]$ and not $[-\pi, \pi]$
  - fftshift transform fft result from $[0, 2\pi]$ to $[-\pi, \pi]$
  - abs absolute value
    - using abs() on the results of Fourier maintains magnitude and discards the phase information

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Example cont (Matlab reference code)

```
function fftExample()
%sampling parameters
fs = 500; % Hz, sampling frequency
timeBinSize = 1/fs;
 %create signal
signalFrequency1 = 15;
signalFrequency2 = 3;
timeRange = 0: timeBinSize :2;
sig1 = 10*sin(2*pi*signalFrequency1*timeRange);
sig2 = 40*cos(2*pi*signalFrequency2*timeRange);
sigTotal = sig1+sig2;
 %analyze signal in frequency domain
t =0: timeBinSize:(length(sigTotal)-1) /fs;
sigTotalF = abs(fftshift(fft(sigTotal)));
freqRange = -fs/2: fs/(length(sigTotal)-1):fs/2; %same length (num of bins) as timeRange
 %display
subplot(2,1,1); plot(t, sigTotal, '-b.', 'LineWidth',2 );
xlabel('t [sec]'); ylabel('amp'); title('time domain'); grid on;
subplot(2,1,2); plot(freqRange, 20*log10(sigTotalF), '-bx', 'LineWidth',2 );
xlabel('freq [Hz]'); ylabel('Power dB'); title('freq domain'); grid on;
 return;
```

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The sampling theorem

- **Nyquist Theorem:** you need 2 samples per “cycle” of your input signal to define it.
- You can accurately measure the frequency of a signal with frequency *f* as long as you are sampling it at greater than 2*f*.
- If you try to measure the frequency of signals having a frequency above *f* with a sampler operating at 2*f*, you will alias the signal, or create false images of this signal at frequencies below *f*.
- These false frequencies will appear as mirror images of the original frequency around the Nyquist frequency. This situation is called "aliasing back" or "folding back"

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Aliasing

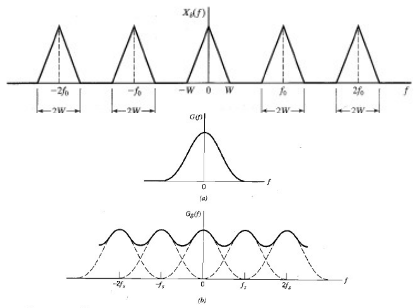

FIGURE 3.3 (a) Spectrum of a signal. (b) Spectrum of an undersampled version of the signal exhibiting the aliasing phenomenon.

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Example exam 2006

The electrical potential generated by the Electrical Frog may be described by the function
V(t) = 1 + X * sin(50*t*2π) + Y * cos(70*t*2π).

a. Assuming that the scientist samples the potential at 120 sample/s, draw the spectrum of the sampled signal – V(ω).

b. Assuming that the sampling rate cannot increase. Provide a solution for extracting X and Y.

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Example 2006

Neuron X fires at a mean rate of 2 spikes/s and its spectrum has a peak around 9Hz and neuron Y fires at a mean rate of 9 spikes/s and its spectrum has a peak around 2Hz.

a. X & Y are possible
b. X & Y are impossible
c. X is possible & Y is not.
d. Y is possible & X is not.

Find fs (sampling frequency) for the possible scenario.

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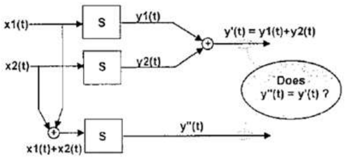
Linear systems



Homogeneous:
- α *f*(*x*) = *f*(α*x*)

Additive:
- *f*(*x* + *y*) = *f*(*x*) + *f*(*y*).

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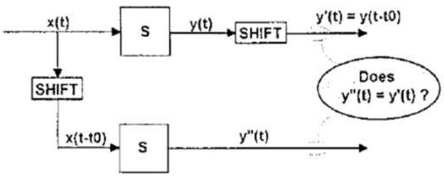
Time invariant systems

- The behavior of the system is fixed over time.



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LTI - Example exam 2007

The amplifier neurons of the Levis Systemis function have the following response function: y(t)=2x(t). The neurons therefore act as a:

a. Linear system.
b. Time invariant system.
c. Linear time invariant (LTI) system.
d. None of the above.

FIR and IIR

- Finite Impulse Response (**FIR**)

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

  - The impulse response fades to zero at a certain point
  - more simple, stable requires higher orders

- Infinite Impulse Response (**IIR**)

$$y[n] = \sum_{k=1}^{N} a_k \cdot y[n-k] + \sum_{k=0}^{M} b_k \cdot x[n-k]$$

  - The impulse response does not fade to zero at any point
  - less simple, sometimes unstable requires lower orders

FIR & IIR basic examples examples

- IIR oscillating impulse response: y(n) = x(n) + -y(n-1)
- IIR exploding impulse response : y(n) = x(n) + 2y(n-1)
- FIR average last 5 samples

Example 2005

Draw the impulse response of a IIR filter defined by:
y(n)=0.5*y(n-1)+x(n).

Calculate an FIR filter which will give equivalent output (with an impulse response error <10%).

Solution

Impulse: 100000000; x(1) = 1; all other x's = 0
I(1) = 0.5*y(0)+x(1) = 1
I(2) = 0.5*y(1) + x(2) = 0.5
I(3) = 0.5*y(2) + x(3) = 0.25
I(4) = 0.5*y(3) + x(4) = 0.125
…
I(n) = $0.5^{\wedge(n-1)}$. A geometric series: a = 1; r = 0.5
Sum(y) = a/(1-r) = 1/0.5 **= 2**;
First 4 terms: I(1)+I(2)+I(3)+I(4) = 1+0.5+0.25+0.125 = 1.875
1.875 > 2*0.9 (<10% error)

Example exam 2007

The filter described by its impulse response y(t)=x(t)+y(t-1):

a. Is a FIR filter. It is possible to create an equivalent IIR filter.
b. Is a FIR filter. It is impossible to create an equivalent IIR filter.
c. Is an IIR filter. It is possible to create an equivalent FIR filter.
d. Is an IIR filter. It is impossible to create an equivalent FIR filter.